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THE LEBESGUE INTEGRAL AND
THE HENSTOCK-KURZWEIL INTEGRAL

Its Relation to
Local Convex Vector Spaces
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THE LEBESGUE INTEGRAL AND
THE HENSTOCK-KURZWEIL INTEGRAL
Its Relation to
Local Convex Vector Spaces

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PREFACE

The approach to integration by Riemannian sums was rehabilitated in the fifties of the 20th century by a new interpretation of the concept of a "fine" or "δ-fine" partition of the integration interval. It is well known that both the Lebesgue integration and the Henstock-Kurzweil integration can be obtained by the same method, only the integration bases are different. The concept of an integration basis \( \mathcal{Y} \) is very flexible and results in a rich class of \( \mathcal{Y} \)-integrations. To every integration basis \( \mathcal{Y} \) there corresponds the vector space \( P_\mathcal{Y} \) of primitives of \( \mathcal{Y} \)-integrable functions (on a fixed interval \( I = [a, b] \subset \mathbb{R} \)), a concept of \( \mathcal{E}_\mathcal{Y} \)-convergent sequence of functions from \( P_\mathcal{Y} \), and \( U_{LC}(\mathcal{E}_\mathcal{Y}) \) which is the finest locally convex topology on \( P_\mathcal{Y} \) such that every \( \mathcal{E}_\mathcal{Y} \)-convergent sequence is convergent in \( (P_\mathcal{Y}, U_{LC}(\mathcal{E}_\mathcal{Y})) \).

Lebesgue integration is obtained by a suitable choice of \( \mathcal{Y}, \mathcal{Y} = \mathcal{L} \). Then \( P_\mathcal{L} \) is the space of absolutely continuous functions and \( U_{LC}(\mathcal{E}_\mathcal{L}) \) is induced by the norm \( \|F\|_{\text{var}} = \text{var} F \). Hence \( (P_\mathcal{L}, U_{LC}(\mathcal{E}_\mathcal{L})) \) is a complete space. If \( \mathcal{Y} = \mathcal{HK} \) then Henstock-Kurzweil integration is obtained: The topology \( U_{LC}(\mathcal{E}_{\mathcal{HK}}) \) is induced by the norm \( \|F\|_{\text{sup}} \) and \( (P_{\mathcal{HK}}, U_{LC}(\mathcal{E}_{\mathcal{HK}})) \) is not complete.

The problem whether \( (P_\mathcal{Y}, U_{LC}(\mathcal{E}_\mathcal{Y})) \) is complete is the central problem of this book. A theory is developed which gives an answer for a broad class of \( \mathcal{Y} \)'s and to an extended problem which includes integrations introduced by Bongiorno and Pfeffer in 1992 and by Bongiorno in 1996.
Topics connected with the Riemann approach to integration were reported and discussed in the Seminar on Differential Equations and Real Functions of the Mathematical Institute of the Academy of Sciences of the Czech Republic since the beginning of this approach.

I wish to thank the participants of the seminar for their contributions and comments.

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Jaroslav Kurzweil
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0 INTRODUCTION

The approach to integration which is based on approximation of the integral by Riemannian sums is rather flexible. If the set of partitions which are used in the formation of Riemannian sums is rich then Lebesgue integration is obtained. On the other end of the spectrum a poor set of partitions leads to an integration which is called Henstock-Kurzweil and which is equivalent to Denjoy integration in the restricted sense and to Perron integration. In this book integrations are studied for various sets of partitions. If \( \mathcal{Y} \) is a set of partitions we denote by \( P_\mathcal{Y} \) the set of primitives of \( \mathcal{Y} \)-integrable functions.

For every \( \mathcal{Y} \) some sequences \( F_i \in P_\mathcal{Y} \) are called \( \mathcal{E}_\mathcal{Y} \)-convergent to a limit \( F_0 \in P_\mathcal{Y} \), \( F_i \xrightarrow{\mathcal{E}_\mathcal{Y}} F_0 \). Therefore there exists a unique locally convex topology \( \mathcal{U}_{LC}(\mathcal{E}_\mathcal{Y}) \) on \( P_\mathcal{Y} \) which is the finest one among locally convex topologies \( \mathcal{T} \) on \( P_\mathcal{Y} \) with the property that \( F_i \overset{\mathcal{E}_\mathcal{Y}}{\to} F_0 \) implies that \( F_i \to F_0 \) in \( (P_\mathcal{Y}, \mathcal{T}) \). The problem whether \( (P_\mathcal{Y}, \mathcal{U}_{LC}(\mathcal{E}_\mathcal{Y})) \) is complete is crucial for this book; the answer depends on \( \mathcal{Y} \).

Let \( I = [a, b] \subset \mathbb{R} \). A set \( \Delta = \{(t_i, A_i); i = 1, 2, \ldots, k\} \) is called a partition in \( I \) if \( k \in \mathbb{N} \) and if

(0.1) \( t_i \in I \) for \( i = 1, 2, \ldots, k \),

(0.2) \( A_i \subset I \) is a figure, i.e. a finite union of closed intervals, \( i = 1, 2, \ldots, k \),

(0.3) figures \( A_i, A_j \) are nonoverlapping for \( i \neq j \) (i.e. \(|A_i \cap A_j| = 0\) for \( i \neq j \) where \(|E|\) is the Lebesgue measure of
$E \subseteq \mathbb{R}$.

$\Delta$ is called a partition of $I$ if (in addition)

\[(0.4) \quad \bigcup_{i=1}^{k} A_i = I.\]

Denote by $\mathcal{L}$ the set of partitions in $I$ and by $\mathcal{HK}$ the subset of $\mathcal{L}$ which consists of $\Delta$ such that

\[(0.5) \quad A_i \text{ is a closed interval, } i = 1, 2, \ldots, k,\]

\[(0.6) \quad t_i \in A_i, \quad i = 1, 2, \ldots, k.\]

Let $\zeta : I \to \mathbb{R}^+$. $\Delta \in \mathcal{L}$ is called $\zeta$-fine if

\[(0.7) \quad A_i \subseteq (t_i - \zeta(t_i), t_i + \zeta(t_i)) \text{ for } i = 1, 2, \ldots, k.\]

$f : I \to \mathbb{R}$ is called $\mathcal{HK}$-integrable ($\mathcal{L}$-integrable, respectively) if there exists $\gamma \in \mathbb{R}$ and for every $\varepsilon \in \mathbb{R}^+$ there exists $\zeta : I \to \mathbb{R}^+$ such that

$$|\gamma - \sum_{i=1}^{k} f(t_i)|A_i|| \leq \varepsilon$$

whenever $\Delta = \{(t_i, A_i); i = 1, 2, \ldots, k\} \in \mathcal{HK}$ ($\Delta \in \mathcal{L}$ respectively) is a $\zeta$-fine partition of $I$. $\gamma$ is called the $\mathcal{HK}$-integral ($\mathcal{L}$-integral respectively) of $f$ over $I$. It is well known that $f$ is $\mathcal{HK}$-integrable if and only if it is Denjoy integrable in the restricted sense over $I$ or equivalently if and only if it is Perron integrable over $I$, and that the two integrals coincide.

On the other hand, $f$ is $\mathcal{L}$-integrable over $I$ if and only if it is Lebesgue integrable over $I$ and again the two integrals coincide. Moreover, two integrations intermediate between $\mathcal{L}$-integration and $\mathcal{HK}$-integration were introduced in [B-Pf 1992] and [B 1996].

In this book integrations intermediate between $\mathcal{L}$-integration and $\mathcal{HK}$-integration are treated systematically. Let $\mathcal{Y}$ fulfil
\(\mathcal{Y} \subset \mathcal{Y} \subset \mathcal{L}\) and some additional conditions which are not important at the moment.

Let \(K \subset I\) be a closed interval. \(f : K \rightarrow \mathbb{R}\) is called \(\mathcal{Y}\)-integrable (over \(K\)) if there exists \(\gamma \in \mathbb{R}\) and for every \(\varepsilon \in \mathbb{R}^+\) there exists \(\zeta : I \rightarrow \mathbb{R}^+\) such that

\[
|\gamma - \sum_{i=1}^{k} f(t_i)|A_i| \leq \varepsilon
\]

whenever \(\Delta = \{(t_i, A_i); i = 1, 2, \ldots, k\} \in \mathcal{Y}\) is a \(\zeta\)-fine partition of \(K\).

If \(f : I \rightarrow \mathbb{R}\) is \(\mathcal{Y}\)-integrable over \(I\) and if \(K \subset I\) is a closed interval then the restriction \(f|_K\) is \(\mathcal{Y}\)-integrable over \(K\). Denote its integral by \(F(K)\). \(F\) maps the set of closed intervals \(K \subset I\) to \(\mathbb{R}\) and is called the primitive of \(f\). If \(F_i, i = 1, 2\) is the primitive of a \(\mathcal{Y}\)-integrable \(f_i : I \rightarrow \mathbb{R}\) then \(F_1 = F_2\) if and only if \(f_1 = f_2\) a.e. In order to avoid the concept of equivalence classes of integrable functions the results will be formulated for the primitives.

Denote by \(P_Y\) the set of \(F\) such that \(F\) is the primitive of some \(\mathcal{Y}\)-integrable \(f\).

A sequence \(F_l \in P_Y, l \in \mathbb{N}\) is called Cauchy in \(E_Y\) if there exists a sequence \(f_l : I \rightarrow \mathbb{R}\) of \(\mathcal{Y}\)-integrable functions fulfilling (0.8) there exist \(\zeta_j : I \rightarrow \mathbb{R}^+\) such that

\[
|F_l(I) - \sum_{i=1}^{k} f_l(t_i)|A_i| \leq 2^{-j}
\]

whenever \(j, l \in \mathbb{N}\) and \(\Delta = \{(t_i, A_i); i = 1, 2, \ldots, k\} \in \mathcal{Y}\) is a \(\zeta_j\)-fine partition of \(I\),

\(0.9\) the sequence \(f_l(t), l \in \mathbb{N}\) is Cauchy in \(\mathbb{R}\) for every \(t \in I\).

Of course, \(F_l\) is the primitive of \(f_l\).

If the sequence \(F_l, l \in \mathbb{N}\) is Cauchy in \(E_Y\) and \(f_l\) fulfil (0.8) and (0.9) then there exists \(f_0 : I \rightarrow \mathbb{R}\) such that

\(0.10\) \(f_l(t) \rightarrow f_0(t)\) for \(l \rightarrow \infty, t \in I\)
and it can be proved that for every closed interval \( K \subset I \) there exists \( F_0(K) \in \mathbb{R} \) and

\[
(0.11) \quad F_l(K) \rightarrow F_0(K) \quad \text{for} \quad l \rightarrow \infty,
\]

that \( f_0 \) is \( \mathcal{Y} \)-integrable and \( F_0 \) is its primitive.

Therefore a convergence \( \mathbb{E}_\mathcal{Y} \) can be introduced on \( P_\mathcal{Y} \) in a natural way as follows.

Let \( F_m \in P_\mathcal{Y} \) for \( m = 0, 1, 2, \ldots \). The sequence \( F_l, \ l \in \mathbb{N} \)
is called convergent to \( F_0 \) in \( \mathbb{E}_\mathcal{Y} \), \( F_l \xrightarrow{\mathbb{E}_\mathcal{Y}} F_0 \) if there exist \( \mathcal{Y} \)-integrable functions \( f_m : I \rightarrow \mathbb{R}, \ m = 0, 1, 2, \ldots \) such that (0.8) and (0.10) hold.

Again, \( F_m \) is the primitive of \( f_m, \ m = 0, 1, 2, \ldots \).

\( \mathbb{E}_\mathcal{Y} \) is the set of couples \( ((F_l, l \in \mathbb{N}), F_0) \) such that \( F_l \xrightarrow{\mathbb{E}_\mathcal{Y}} F_0 \).

A topology \( T \) on \( P_\mathcal{Y} \) is called tolerant to \( \mathbb{E}_\mathcal{Y} \) if \( F_l \xrightarrow{T} F_0 \) implies that \( F_l \rightarrow F_0 \) in \((P_\mathcal{Y}, T)\).

It follows from general considerations that there exists a locally convex topology \( \mathcal{U}_{LC}(\mathbb{E}_\mathcal{Y}) \) on \( P_\mathcal{Y} \) such that

\[
(0.12) \quad \mathcal{U}_{LC}(\mathbb{E}_\mathcal{Y}) \text{ is tolerant to } \mathbb{E}_\mathcal{Y},
\]

\[
(0.13) \quad \mathcal{U}_{LC}(\mathbb{E}_\mathcal{Y}) \text{ is finer than any locally convex topology } T \text{ on } P_\mathcal{Y} \text{ which is tolerant to } \mathbb{E}_\mathcal{Y}.
\]

Obviously, \( \mathcal{U}_{LC}(\mathbb{E}_\mathcal{Y}) \) is unique. It was proved in [K 2000] that \( \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{H}K}) \) is the topology on \( P_{\mathcal{H}K} \) induced by the norm \( \| \cdot \|_{sup} \) where

\[
\| F \|_{sup} = \sup \{|F(K)|; \ K \subset I \text{ is a closed interval } \}.
\]

Therefore \((P_{\mathcal{H}K}, \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{H}K}))\) is not complete. In this book the answer to the problem of completeness of \((P_\mathcal{Y}, \mathcal{U}_{LC}(\mathbb{E}_\mathcal{Y}))\) is given for various \( \mathcal{Y} \).

The topics of Chapters 1-3 are clear from their headings. In Chapter 2 convergence \( Q_\mathcal{Y} \) is introduced in addition to the convergence \( \mathbb{E}_\mathcal{Y} \).

A system of sets \( Q_\mathcal{Y}(\delta) \subset P_\mathcal{Y} \) is defined where \( \delta \) is a parameter, \( \delta \in D^* \).
Every set $Q_Y(\delta)$ is convex, circled and compact in $P_Y$ which is endowed with the topology induced by $\| \cdot \|_{\text{sup}}$. Moreover,

$$P_Y = \bigcup_{\delta \in D^*} Q_Y(\delta).$$

A sequence $F_l \in P_Y$, $l \in \mathbb{N}$ is called convergent to $F_0$ in $Q_Y$, $F_l \xrightarrow{Q_Y} F_0$ (where $F_0 \in P_Y$), if there exists $\delta \in D^*$ such that $F_l \in Q_Y(\delta)$ for $l \in \mathbb{N}$ and if $\|F_l - F_0\|_{\text{sup}} \to 0$ for $l \to \infty$.

$Q_Y$ is the set of couples $((F_l, l \in \mathbb{N}), F_0)$ such that $F_l \xrightarrow{Q_Y} F_0$.

Elementary relations between convergence and topology are studied in Chapter 3.

A topology $T$ on $P_Y$ is called tolerant to $Q_Y$ if $F_l \xrightarrow{Q_Y} F_0$ implies that $F_l \to F_0$ in $(P_Y, T)$.

Again, there exists a unique locally convex topology $U_{LC}(Q_Y)$ on $P_Y$ such that

(0.14) $U_{LC}(Q_Y)$ is tolerant to $Q_Y$,

(0.15) $U_{LC}(Q_Y)$ is finer than any locally convex topology $T$ on $P_Y$ which is tolerant to $Q_Y$.

The convergences $E_Y$ and $Q_Y$ are so close that

(0.16) a topology $T$ on $P_Y$ is tolerant to $E_Y$ if and only if it is tolerant to $Q_Y$.

Consequently,

(0.17) $U_{LC}(E_Y) = U_{LC}(Q_Y)$.

As a consequence of (0.14) and (0.15) we find that

(0.18) the topology which is induced on $Q_Y(\delta)$ by $\| \cdot \|_{\text{sup}}$ is finer than the restriction of $U_{LC}(Q_Y)$ to $Q_Y(\delta)$, $\delta \in D^*$;

(0.19) if $T$ is a locally convex topology on $P_Y$ such that for every $\delta \in D^*$ the topology which is induced on $Q_Y(\delta)$ by $\| \cdot \|_{\text{sup}}$ is finer than the restriction of $T$ to $Q_Y(\delta)$, then $U_{LC}(Q_Y)$ is finer than $T$. 
Finally, it can be deduced from (0.18) and (0.19) that

\[(0.20) \quad \{\text{conv } \bigcup_{\delta \in D^*} Q_Y(\delta) \cap B(\eta(\delta)); \eta : D^* \to \mathbb{R}^+\}\]

is a zero-filterbase for \(U_{LC}(Q_Y)\). In the above formula

\[B(\sigma) = \{F \in P_Y; \|F\|_{\text{sup}} < \sigma\}, \sigma \in \mathbb{R}^+\]

and \(\text{conv}E\) is the convex hull of \(E \subset P_Y\). (0.20) holds for all \(Y\).

In Chapter 4 some restrictions on \(Y\) are introduced and seminorms \(\| \cdot \|_{Y,s}, s \in I\) are found in an explicit form such that

\[(0.21) \quad U_{LC}(Q_Y)\text{ is the topology on } P_Y\text{ induced by the set of seminorms }\| \cdot \|_{Y,s}, s \in I\]

(cf. Theorem 4.2). In the proof of (0.21) a major part is played by (0.20).

In Section 5 the case \(Y = \mathcal{L}\) is treated. Since the restrictions from Section 4 are fulfilled in the case \(Y = \mathcal{L}\) and since for \(s \in I\) the seminorm \(\| \cdot \|_{\mathcal{L},s}\) is equivalent to \(\| \cdot \|_{\text{var}}\) where \(\|F\|_{\text{var}} = \text{var} F\), we conclude that \(U_{LC}(Q_{\mathcal{L}})\) is induced on \(P_{\mathcal{L}}\) by the norm \(\| \cdot \|_{\text{var}}\).

Finally, it is deduced that \(P_{\mathcal{L}}\) is the set of primitives of Lebesgue integrable functions. \(\mathcal{L}\)-integration coincides with Lebesgue integration and \((P_{\mathcal{L}}, U_{LC}(Q_{\mathcal{L}}))\) is complete.

Observe that \(F_l \xrightarrow{Q_{\mathcal{L}}} F_0\) implies that \(\|F_l - F_0\|_{\text{var}} \to 0\) for \(l \to \infty\) but not vice versa.

The \(\mathcal{M}\)-integration is close to \(\mathcal{L}\)-integration and \((P_{\mathcal{M}}, U_{LC}(Q_{\mathcal{M}}))\) is complete (cf. Chapter 6).

In Section 7 some new restrictions on \(Y\) are introduced which together with the restrictions from Section 4 guarantee that \((P_Y, U_{LC}(Q_Y))\) is not complete.

Let \(\Lambda\) be the set of \(\lambda : [0, \infty) \to [0, \infty)\) nondecreasing, \(\lambda(0) = 0, \lambda(\sigma) > 0\) for \(\sigma > 0\).
Let \( S(\lambda) \) be the set of partitions \( \Delta = \{ (t_i, J_i); i = 1, 2, \ldots, k \} \) in \( I \) such that \( k \in \mathbb{N}, J_i \) is a closed interval for \( i = 1, 2, \ldots, k \) and
\[
\sum_{i=1}^{k} \lambda(\text{dist}(t_i, J_i)) \leq 1
\]
(dist\( (t, J) \) being the distance of \( t \) from \( J \)).

Then \( (P_{S(\lambda)}, U_{LC}(\mathbb{Q}_{S(\lambda)})) \) is not complete since the restrictions from Sections 4 and 7 are fulfilled by \( S(\lambda) \) for \( \lambda \in \Lambda \) (Chapter 8).

Let \( \Omega \) be the set of \( \omega : \mathbb{R}^+ \rightarrow [0, \frac{1}{2}], \) nondecreasing, \( \omega(\sigma) > 0 \) for \( \sigma > 0 \). Let \( \mathcal{R}(\omega) \) be the set of partitions \( \Delta = \{ (t_i, A_i); i = 1, 2, \ldots, k \} \) in \( I \) such that \( k \in \mathbb{N} \) and
\[
\frac{|A_i|}{\|A_i\| \text{diam}\{t_i\} \cup A_i} \geq \omega(\text{diam}\{t_i\} \cup A_i), \ i = 1, 2, \ldots, k
\]
where \( \text{diam} E \) is the diameter of \( E \subset \mathbb{R} \) and \( \|A\| \) is twice the number of the components of the figure \( A \). Then
\[
(P_{\mathcal{R}(\omega)}, U_{LC}(\mathbb{Q}_{\mathcal{R}(\omega)})) \text{ is not complete}
\]
(Chapter 9).

Let \( \mathcal{X} = (\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots), f : I \rightarrow \mathbb{R} \). \( f \) is called \( \mathcal{X} \)-integrable and \( F \) is its primitive if \( f \) is \( \mathcal{Y}_k \)-integrable and \( F \) is its primitive for \( k \in \mathbb{N} \). Denote by \( P_{\mathcal{X}} \) the set of \( F \) such that \( F \) is the primitive of some \( \mathcal{X} \)-integrable \( f \). Let \( F_m \in P_{\mathcal{X}} \) for \( m = 0, 1, 2, \ldots \).

The space \( P_{\mathcal{X}} \) is equipped with the topology \( \mathcal{U}_{\mathcal{X}} \) which is defined as the supremum of the set of topologies \( \mathcal{U}_{LC}(\mathbb{Q}_{\mathcal{Y}_k}), k \in \mathbb{N} \). The concept of \( \mathcal{X} \)-integration is an extension of the concept of \( \mathcal{Y} \)-integration and is studied in Chapter 10. Noncompleteness results are obtained if \( \mathcal{X} = (\mathcal{S}(\lambda_1), \mathcal{S}(\lambda_2), \mathcal{S}(\lambda_k), \ldots), \lambda_k \in \Lambda \) for \( k \in \mathbb{N} \) and if \( \mathcal{X} = (\mathcal{R}(\omega_1), \mathcal{R}(\omega_2), \mathcal{R}(\omega_k), \ldots), \omega_k \in \Omega \) for \( k \in \mathbb{N} \). If \( \lambda_k(\sigma) = 2^k \sigma \) for \( \sigma \geq 0, \mathcal{X} = (\mathcal{S}(\lambda_1), \mathcal{S}(\lambda_2), \mathcal{S}(\lambda_k), \ldots) \) then \( \mathcal{X} \)-integration is the integration which was introduced in \([B \)
1996], if \( \omega_k(\sigma) = 2^{-k} \) for \( \sigma \geq 0 \), \( \mathcal{X} = (\mathcal{R}(\omega_1), \mathcal{R}(\omega_2), \mathcal{R}(\omega_k), \ldots) \) then \( \mathcal{X} \)-integration is the integration which was studied in [B-Pf 1992]. Noncompleteness results in Chapters 9 and 10 are obtained by comparison with \( S(\lambda) \)-integration for a suitable \( \lambda \in \Lambda \).

\( \mathcal{Y} \)-differentiation is introduced in Chapter 11 and a general result concerning the relation of \( \mathcal{Y} \)-differentiation and \( \mathcal{X} \)-integration is obtained. Chapter 11 is concluded by a specialization to \( \mathcal{L} \)-integration.

This Chapter will be closed by a list of four parts of the book which are not necessary for understanding the text which follows them:

1. Sections 4.7, 4.8, pp. 49–51 (a necessary and sufficient condition for completeness of \( (P_Y, U_{LC}(Q_Y)) \) for a subclass of \( \mathcal{Y} \)'s);

2. Sections 5.8 – 5.11, pp. 62–68 (relations between convergences in \( (P_L, U_{LC}(Q_L)) \), in \( Q_L \) and in \( E_L \));

3. Chapter 6, pp. 69–75 (\( \mathcal{M} \)-integration);

4. Sections 8.9 – 8.12, pp. 91–103 (dependence of \( S(\lambda) \)-integration on \( \lambda \)).
1 BASIC CONCEPTS AND 
PROPERTIES OF γ-INTEGRATION

1.1 Notation. By \( \mathbb{R}, \mathbb{R}^+, \mathbb{N}, \mathbb{Z} \) we denote the set of reals, the set of positive reals, the set of positive integers, the set of integers. For \( E \subseteq \mathbb{R}, t \in \mathbb{R} \) let \( \text{int} E, \text{cl} E, |E|, \text{diam} E, \text{dist}(t, E) \) be the interior of \( E \), the closure of \( E \), the outer Lebesgue measure of \( E \), the diameter of \( E \), the distance of \( t \) from \( E \).

Let \( \mathcal{N} \) be the set of \( N \subseteq \mathbb{R} \) such that \( |N| = 0 \). By intervals we mean compact nondegenerate intervals in \( \mathbb{R} \), e.g. \( L = [c, d] \). An open interval is denoted by \( (c, d) \) while \( [c, d), (c, d] \) are semiopen intervals. A figure is a finite union of intervals. The symbol \( \#M \) stands for the number of elements of a finite set \( M \).

Let \( I = [a, b] \) be an interval. If \( K \subseteq I \), then \( \text{Iv}(K) \) is the set of intervals \( L \subseteq K \) and \( \text{Fig}(K) \) is the set of figures \( A \subseteq K \). We shall write \( \text{Iv}, \text{Fig} \) instead of \( \text{Iv}(I), \text{Fig}(I) \).

Let \( K \subseteq I \). A finite set \( \Delta = \{(t_i, A_i); i = 1, 2, \ldots, k\} \), shortly \( \Delta = \{(t, A)\} \) is called a partition in \( K \) if

(1.1) \( t_i \in I, A_i \in \text{Fig}, A_i \subseteq K \) for \( i = 1, 2, \ldots, k \)

and

(1.2) \( |A_i \cap A_j| = 0 \) whenever \( (t_i, A_i), (t_j, A_j) \in \Delta, i \neq j \).

\( \Delta \) is called a partition of \( K \) if in addition

\[ K = \bigcup \{A_i; i = 1, 2, \ldots, k\}. \]
Let $\zeta : I \to \mathbb{R}^+$, $E \subset I$. A partition $\Delta = \{(t, A)\}$ is called

(1.3) $\zeta$-fine, if $A \subset [t - \zeta(t), t + \zeta(t)]$ for $(t, A) \in \Delta$,

(1.4) $E$-tagged, if $t \in E$ for $(t, A) \in \Delta$ ($t$ is the tag of $A$).

Let $\mathcal{L}$ be the set of partitions in $I$ and let $\mathcal{H}_\mathcal{K}$ be the set of partitions $\Delta$ in $I$ such that $(t, A) \in \Delta \in \mathcal{H}_\mathcal{K}$ implies

$$t \in A \in \mathcal{I}_v.$$

By $\mathcal{Y}$ we denote a set of partitions $\Delta$ in $I$ such that

(1.5) $\mathcal{H}_\mathcal{K} \subset \mathcal{Y} \subset \mathcal{L}$,

(1.6) if $\Theta \subset \Delta \in \mathcal{Y}$, then $\Theta \in \mathcal{Y}$,

(1.7) if $\Delta = \{(t_i, A_i); i = 1, 2, \ldots, k\} \in \mathcal{Y}, \Theta = \{(s_j, L_j); j = 1, 2, \ldots, l\} \in \mathcal{H}_\mathcal{K}$, $|A_i \cap L_j| = 0$ for $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$ then

$$\Delta \cup \Theta \in \mathcal{Y}.$$

The set of $\mathcal{Y}$'s is denoted by $\mathcal{Y}$. Throughout the book it will be assumed that

$$\mathcal{Y} \in \mathcal{Y}.$$

Let $E \subset M \subset I$, $\zeta : I \to \mathbb{R}^+$. By $\mathcal{Y}(M, E, \zeta)$ we denote the set of $\Delta = \{(t, A)\} \in \mathcal{Y}$ such that $\Delta$ is a partition in $M$ which is $E$-tagged and $\zeta$-fine.

Occasionally $\zeta$ is independent of $t$, i.e. there exists $\alpha \in \mathbb{R}^+$ and $\zeta(t) = \alpha$ for $t \in I$, and we write $\mathcal{Y}(M, E, \alpha)$ instead of $\mathcal{Y}(M, E, \zeta)$ in that case. Moreover, $\mathcal{Y}(M, E, \infty)$ is the set of $\Delta = \{(t, A)\} \in \mathcal{Y}$ such that $t \in E, A \subset M$.

The set of $\delta : \mathbb{N} \times I \to \mathbb{R}^+$ such that

$$\delta(j, t) \geq \delta(j + 1, t) \text{ for } j \in \mathbb{N}, t \in I$$

is denoted by $D^*$. $\delta(j, \cdot) : I \to \mathbb{R}^+$ is defined by $\delta(j, \cdot)(t) = \delta(j, t)$ for $j \in \mathbb{N}, t \in I$.

If $f : I \to \mathbb{R}, M \subset I, F : \text{Fig} \to \mathbb{R}$ then $f|_M, F|_{\text{Fig}(M)}$ are the restrictions of $f$ to $M$ and of $F$ to $\text{Fig}(M)$; $F|_{\text{lv}(M)}$ has an analogous meaning. If $\delta \in D^*$ then $\delta|_M$ is the restriction of $\delta : \mathbb{N} \times I \to \mathbb{R}^+$ to $\mathbb{N} \times M$, i.e. $\delta|_M(j, \cdot) : M \to \mathbb{R}^+$ for $j \in \mathbb{N}$.
1.2 Lemma (Cousin). Let $K \in \mathcal{I}$, $\zeta : I \to \mathbb{R}^+$. Then there exists $\Delta = \{(t, A)\} \in \mathcal{HC}(I, I, \zeta)$, $\Delta$ being a partition of $K$.

The lemma is well known (cf. e.g. [K 2000], Lemma 1.2). Taking (1.7) into account we have

Corollary. If $\zeta : I \to \mathbb{R}^+$, $\Theta \in \mathcal{Y}(I, I, \zeta)$ then there exists $\Delta \in \mathcal{Y}(I, I, \zeta)$ such that $\Delta$ is a partition of $I$ and $\Theta \subseteq \Delta$.

1.3 Definition. Let $K \in \mathcal{I}$, $f : K \to \mathbb{R}$. $f$ is called $\mathcal{Y}$-integrable on $K$ if

(1.9) there exists $\gamma \in \mathbb{R}$, $\delta \in D^*$ such that

$$|\gamma - \sum_{\Delta} f(t)|A|| = |\gamma - \sum_{i=1}^{k} f(t_i)|A_i|| \leq 2^{-j}$$

for $j \in \mathbb{N}$, $\Delta = \{(t, A)\} = \{(t_i, A_i); i = 1, 2, \ldots, k\} \in \mathcal{Y}(K, K, \delta(j, \cdot))$, $\Delta$ being a partition of $K$.

Notes. (i) The value $\gamma$ is unique by Lemma 1.2. It is called the $\mathcal{Y}$-integral of $f$ over $K$ and denoted by $(\mathcal{Y}) \int_{K} f dt$. It follows from (1.5) that every $\mathcal{Y}$-integrable $f$ is $\mathcal{HC}$-integrable and that

$$(\mathcal{Y}) \int_{K} f dt = (\mathcal{HC}) \int_{K} f dt.$$

Therefore we will usually write $\int_{K} f dt$ instead of $(\mathcal{Y}) \int_{K} f dt$.

(ii) $\mathcal{HC}$-integrable means Henstock-Kurzweil integrable (cf. [K 2000], Definition 1.4), $\mathcal{L}$-integrable means Lebesgue integrable (see Chapter 5).

1.4 Theorem. Let $f : I \to \mathbb{R}$ be $\mathcal{Y}$-integrable on $I$, $K \in \mathcal{I}$. Then

(1.10) the restriction $f|_{K}$ is $\mathcal{Y}$-integrable on $K$. 
$F : \text{Fig} \to \mathbb{R}$ is defined by

$$F(K) = (\mathcal{Y}) \int_K f dt = (\mathcal{Y}) \int_K f|_K dt$$

and is called the primitive of $f$ ($\mathcal{Y}$-primitive of $f$).

**Proof.** Let $\delta \in D^*$ correspond to $f$ by Definition 1.3,

$$K = \bigcup_{k=1}^l [c_k, d_k]$$

where $l \in \mathbb{N}$ and

$$a \leq c_1 < d_1 < c_2 < d_2 < \cdots < c_l < d_l \leq b.$$

Let $L_m, m \in \mathbb{M}$ be the components of $I \setminus K$, $(l - 1 \leq \# M \leq l + 1)$. Let $\Delta = \{(t, A)\}, \Delta' = \{(t', A')\} \in \mathcal{Y}(K, K, \delta(j + 1, \cdot))$ be partitions of $K$. By Lemma 1.2 there exists $\Theta_m \in \mathcal{H} \mathcal{K}(\text{cl} L_m, \text{cl} L_m, \delta(j + 1, \cdot))$ for $m \in \mathbb{M}$. Put

$$\Xi_1 = \Delta \cup \bigcup_{m \in \mathbb{M}} \Theta_m, \quad \Xi_2 = \Delta' \cup \bigcup_{m \in \mathbb{M}} \Theta_m.$$

Both $\Xi_1$ and $\Xi_2$ are $\delta(j + 1, \cdot)$-fine partitions of $I$ so that

$$|F(I) - \sum_{t \in \Xi_i} f(t)|A|| \leq 2^{-j-1}, \; i = 1, 2$$

and

$$|\sum_{t \in \Xi_1} f(t)|A| - \sum_{t \in \Xi_2} f(t)|A|| \leq 2^{-j},$$

which implies that $f|_K$ is $\mathcal{Y}$-integrable on $K$. The proof is complete.
1.5 **Definition.** \( H : \text{Fig} \to \mathbb{R} \) is called *additive* if

\[
H(K_1 \cup K_2) = H(K_1) + H(K_2)
\]

for \( K_1, K_2 \in \text{Fig}, |K_1 \cap K_2| = 0. \)

**Note.** For \( H : \text{Fig} \to \mathbb{R} \) put \( S(a) = 0, S(t) = H([a,t]) \) for \( a < t \leq b. \) Then \( H \) is additive if and only if \( H([c,d]) = S(d) - S(c) \) for \( a < c < d < b \) and \( H(K) = \sum_{i=1}^{r} H(L_i) \) where \( K \in \text{Fig} \) and \( L_1, L_2, \ldots, L_r \) are the components of \( K. \)

1.6 **Theorem.** Let \( f : I \to \mathbb{R} \) be \( \mathcal{Y} \)-integrable and let \( F : \text{Fig} \to \mathbb{R} \) be its primitive. Then \( F \) is additive.

**Proof.** Let \( K_1, K_2 \in \text{Fig}, |K_1 \cap K_2| = 0, j \in \mathbb{N}. \) Let \( \delta \in D^* \) correspond to \( f \) by Definition 1.3. There exist \( \Delta_i = \{(t, A)\} \in \mathcal{Y}(K_i, K_i, \delta(j, \cdot)), \Delta_i \) being a partition of \( K_i, i = 1, 2. \) Then

\[
|F(K_i) - \sum_{\Delta_i} f(t)|A|| \leq 2^{-j}, \ i = 1, 2.
\]

Moreover, \( \Delta_1 \cup \Delta_2 \in \mathcal{Y}(K_1 \cup K_2, K_1 \cup K_2, \delta(j, \cdot)) \) and \( \Delta_1 \cup \Delta_2 \) is a partition of \( K_1 \cup K_2 \) so that

\[
|F(K_1 \cup K_2) - \sum_{\Delta_1 \cup \Delta_2} f(t)|A|| \leq 2^{-j}.
\]

(1.11) and (1.12) imply that

\[
|F(K_1 \cup K_2) - F(K_1) - F(K_2)| \leq 3 \cdot 2^{-j}.
\]

\( F \) is additive since \( j \in \mathbb{N} \) is arbitrary. The proof is complete.